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Cosmic time functions in certain Robinson–Trautman space–times

Bengt Månsson

Department of Theoretical Physics, University of Lund, Lund, Sweden

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Abstract. A class of Robinson–Trautman space–times is investigated with respect to the existence and properties of cosmic time functions and corresponding surfaces of simultaneity, partial Cauchy surfaces (PCS). The existence of an horizon-like hypersurface $\mathcal{N}$ generated by null curves is established. Then it is proved that in any connected PCS, intersecting $\mathcal{N}$ and approaching the curvature singularity in a certain way, the singularity appears as a point. Some counterexamples are given when these conditions do not hold, and the relation to the problem of representing particles by means of singularities is briefly discussed.

1. Introduction

In general relativity physical space–time, i.e. the collection of all events, is represented by a four-dimensional C^∞ manifold $\mathcal{M}$ with Lorentz metric g (Hawking and Ellis 1974, ch 3). The description of a physical course of events then amounts to describing the appearance of a one-parameter family of surfaces of simultaneity (Misner *et al* 1973, ch 27). Therefore, the dynamical object of general relativity is three-dimensional space (Wheeler *et al* 1962, Wheeler 1968).

To make these ideas more precise, one introduces the concept of a *cosmic time function* (CTF), defined as a function t on $\mathcal{M}$, whose gradient is everywhere time-like (Hawking and Ellis 1974, ch 6). Thus, t is increasing along every future directed time-like or null curve. The level surfaces $\{t = \text{constant}\}$ are *partial Cauchy surfaces* (PCS), i.e. space-like hypersurfaces with no edge which no time-like or null curve intersects more than once (Hawking 1968). Any CTF can be taken as a generalisation of the globally defined time of non-relativistic and special-relativistic theories, and the corresponding PCS's then make precise the concept of surfaces of simultaneity in space–time.

Not every space–time manifold admits a CTF; the necessary and sufficient condition for the existence of a CTF has been found to be stable causality (Hawking 1968, Hawking and Ellis 1974). Given this condition the surfaces $\{t = \text{constant}\}$ are, of course, not unique, and in general the corresponding descriptions of the evolution of a physical system are widely different (Misner *et al* 1973, ch 31), although certain choices of t may appear more natural or can be more convenient to use.

In the present paper we investigate CTF's in certain Robinson–Trautman (RT) space–times, which have been used in connection with the problem of motion of particles represented by space–time singularities. These space–times are described and some useful properties are found in § 2. In § 3 we give examples of CTF's whose

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properties seem natural in connection with the description of particles. These belong to a class of CTF's such that the curvature singularity appears as a point in the corresponding PCS's, sufficient conditions for which are given in § 4. Finally, the relation of our results to the problem of representing matter by space–time singularities is discussed briefly in § 5.

2. RT space–times

The line element of the RT space–times can be given in the form (Newman and Posadas 1969, Carmeli 1977, ch 11)

$$ds^2 = 2\left(K - \frac{\dot{P}}{P}r - \frac{M}{r}\right) du^2 + 2\, dr\, du - \tfrac{1}{2}r^2 P^{-2}\, d\zeta\, d\bar{\zeta} \tag{2.1}$$

where

$$K = 4P^2 \partial_\zeta \partial_{\bar{\zeta}}(\ln P). \tag{2.2}$$

Further,

$$\dot{M} - 3M\frac{\dot{P}}{P} = 4P^2 \partial_\zeta \partial_{\bar{\zeta}} K, \tag{2.3}$$

which is equivalent with the vacuum field equations, $R_{ab} = 0$†, for any metric of the form (2.1), (2.2), assuming that P is independent of r and that M is a function of u only. The fourth coordinate $x^4 = u$ is a null coordinate ($g^{44} = 0$) taking all real values, $r > 0$, and ζ takes all complex values. Instead of ζ we will also use polar coordinates defined from $\zeta = e^{i\phi} \cot \frac{1}{2}\theta$, $0 < \theta \leq \pi$, $0 \leq \phi < 2\pi$, and the coordinates are then numbered $(x^a) = (x^1, x^2, x^3, x^4) = (r, \theta, \phi, u)$. The case when the submanifolds $\mathscr{S}(r, u) = \{r, u = \text{constants}\}$ are diffeomorphic to the unit sphere S^2 is selected by including points with $\theta = 0$.

These metrics are functionally form invariant (d'Inverno and Smallwood 1978) under the coordinate transformations (Carmeli 1977, ch 11)

$$u' = s(u) \qquad r' = r\dot{s}^{-1} \qquad \zeta' = f(\zeta) \tag{2.4}$$

where a dot denotes differentiation with respect to u, and f is analytic, with the following transformations of the functions appearing in the metric:

$$M' = M\dot{s}^{-3} \qquad K' = K\dot{s}^{-2} \qquad P' = P\dot{s}^{-1}|\partial_\zeta f|. \tag{2.5}$$

The relation (2.2), as well as the field equation (2.3), then also holds for the transformed quantities. This works *locally* with arbitrary s, f but if a transformation (2.4), (2.5) is required in the whole space–time, with the primed coordinates taking the same values as the unprimed ones, certain restrictions are necessary, e.g. f must be of the form $f(\zeta) = (a\zeta + b)/(c\zeta + d)$, $ad - bc \neq 0$ (Cartan 1963).

Every RT space–time admits a globally defined tetrad $\ell, n, m, \bar{m}$, where ℓ and n are real null vectors‡. We will be particularly interested in n, whose contravariant

† Italic indices take the values 1, 2, 3, 4 and Greek indices 1, 2, 3.
‡ In fact, most derivations of the RT metrics start from such a tetrad, e.g. that of Carmeli (1977).

components can be taken as

$$n^a = \delta^a{}_4 - \tfrac{1}{2} g_{44} \delta^a{}_1 \tag{2.6}$$

in any coordinate system defined by (2.4).

Finally, the RT space–times have a *curvature singularity* at $r = 0$, unless $M \equiv 0$, as is seen from

$$R_{abcd} R^{abcd} = 48 M^2 r^{-6}. \tag{2.7}$$

This expression can be found by direct calculation or, more easily, by making use of the connection with the tetrad components Ψ_A of the Weyl tensor. In empty space

$$R_{abcd} R^{abcd} = 24(\Psi_2{}^2 + \bar{\Psi}_2{}^2) + 8(\Psi_0 \Psi_4 + \bar{\Psi}_0 \bar{\Psi}_4) - 32(\Psi_1 \Psi_3 + \bar{\Psi}_1 \bar{\Psi}_3) \tag{2.8}$$

(cf Campbell and Wainwright 1977) and for RT space–times $\Psi_0 = \Psi_1 = 0$, $\Psi_2 = -Mr^{-3}$ (Carmeli 1977, ch 11).

Those RT space–times which we are considering, i.e. with the $\mathscr{S}(r, u)$ diffeomorphic to the unit sphere S^2, are, so to speak, deformations of Finkelstein's outgoing space–time, and have been of some interest in connection with the problem of motion and structure of particles (Newman and Posadas 1969, Månsson 1978). Equivalently, these space–times can be selected by requiring the so-called fundamental two-dimensional surfaces $F2S = \mathscr{S}(1, u)$ to be diffeomorphic to S^2.

It can be shown (Månsson 1978) that

$$P = P_0 H \tag{2.9}$$

where

$$P_0 = \tfrac{1}{2}(1 + \zeta\bar{\zeta}) = \tfrac{1}{2} \sin^{-2} \tfrac{1}{2}\theta \tag{2.10}$$

and H can be expanded in spherical harmonics. To prevent singularities in $F2S$ it is necessary that $H \neq 0$, and we can choose $H > 0$. From (2.2), (2.9) and (2.10),

$$K = H^2(1 + \eth_0 \eth_0^* (\ln H)) \tag{2.11}$$

where $\eth_0 \eth_0^* = 4P_0{}^2 \partial_\zeta \partial_{\bar{\zeta}}$ is minus the square of the angular momentum operator. Since $H > 0$ and $F2S$ is compact it is clear that K, half the Gaussian curvature of $F2S$, as well as H, can be expanded in spherical harmonics.

In order not to deviate too much from the spherically symmetric case we further assume

(i) as $u \to +\infty$, $F2S$ has the limiting configuration of a sphere, i.e. $K \to$ constant;

(ii) for all (θ, ϕ) and large negative u, $\alpha < H$, $K < \beta$, where α, β are some positive numbers;

(iii) M is a positive constant.

For this subclass of RT space–times we will now establish a theorem concerning the vector field n, to be used in our study of PCS's and CTF's.

Remark. In a linear approximation (Newman and Posadas 1969) $F2S$ approaches a sphere exponentially as $u \to +\infty$. Further, when $\dot{M} = 0$, the field equation (2.3) gives

$$\frac{\mathrm{d}}{\mathrm{d}u} \int H^{-2}\, \mathrm{d}\Omega = \frac{2}{3M} \int \eth_0 \eth_0^* K\, \mathrm{d}\Omega. \tag{2.12}$$

Here $\eth_0 \eth_0^* K$ can be expanded in spherical harmonics with $l \neq 0$ so that the integral on the right vanishes and $F2S$ has constant area.

Theorem 1. Let $\mathscr{M}$ be an RT space–time with $F2S$ diffeomorphic to S^2 and fulfilling conditions (i)–(iii). Then, as $u \to +\infty$, each integral curve of $\boldsymbol{n}$ which is incomplete at the curvature singularity goes asymptotically towards a unique integral curve of $\boldsymbol{n}$, defined for all u and with bounded r coordinate. These asymptotes generate a non-space-like hypersurface with topology $S^2 \times R$.

Proof. The main idea is to perform a coordinate transformation $r \to r'$ such that $n'^a = \delta^a{}_4$ on a hypersurface with constant r'.

First, let

$$K_0 = \lim_{u\to+\infty} K \qquad H_0 = \lim_{u\to+\infty} H \tag{2.13}$$

for any (θ, ϕ). By condition (i) K_0 and, with a suitable ζ coordinate (2.4), H_0 are constants and then (2.11) gives

$$K_0 = H_0{}^2. \tag{2.14}$$

The radial coordinate is transformed according to†

$$r' = rH^{-1}\psi^{-1} \tag{2.15}$$

where ψ is a function of (u, θ, ϕ) satisfying

$$\dot{\psi} = b\psi^{-1} - a \tag{2.16}$$

with

$$a = \tfrac{1}{2}KM^{-1}H^{-1} \qquad b = \tfrac{1}{4}M^{-1}H^{-2} \tag{2.17}$$

such that $\psi > 0$ is defined and bounded for all u, and $\psi \to \frac{1}{2}H_0^{-3}$ as $u \to +\infty$‡. This transformation, applied to (2.6), gives

$$n'^a = \delta^a{}_4 + n'^1\delta^a{}_1 \tag{2.18}$$

where

$$n'^1 = \frac{\partial r'}{\partial u} - \frac{\partial r'}{\partial r}\left(K - \frac{\dot{P}}{P}r - \frac{M}{r}\right) = -\left(1 - \frac{2M}{r'}\right)(r'\dot{\psi} + \tfrac{1}{2}\psi^{-1}H^{-2})\psi^{-1}. \tag{2.19}$$

Thus,

$$n'^a(r' = 2M) = \delta^a{}_4, \tag{2.20}$$

which implies that the curves $\{r' = 2M, \theta = \theta_0, \phi = \phi_0\}$, with any constants (θ_0, ϕ_0), are integral curves of the vector field $\boldsymbol{n}$ defined for all u. From the properties of $F2S$ it is clear that these curves generate a three-dimensional manifold $\mathscr{N}$ with topology $S^2 \times R$. In the original coordinates $\mathscr{N}$ is given by $r = r_{\mathscr{N}}(u, \theta, \phi)$ where

$$r_{\mathscr{N}} \to M/K_0 \qquad \text{as } u \to +\infty \tag{2.21}$$

and $r_{\mathscr{N}}$ is bounded when $u \to -\infty$, so that there exist numbers r_+, r_- such that $0 < r_- < r_{\mathscr{N}} < r_+$ everywhere on $\mathscr{N}$.

† We try to factorise n'^1 with one factor containing no term proportional to r'. A somewhat similar idea was used by Newman and Unti (1962) and by Foster and Newman (1967).

‡ The existence and uniqueness of such a function has been proved, in the main, by L Gårding (1978, private communication). In particular, uniqueness is guaranteed by condition (ii).

Let γ be an integral curve of $\mathcal{N}$ starting at $r=0$, $u=u_0$, $(\theta,\phi)=(\theta_0,\phi_0)$. Since by (2.18) $n'^4=1$, $n'^2=n'^3=0$, u increases strictly along γ, and γ is for all $u>u_0$ given by $r'=r'_\gamma(u)$, where $0<r'_\gamma(u)<2M$. Now, suppose $r'_\gamma(u)<2M-\epsilon$ with some $\epsilon>0$. Then, by (2.19), $n'^1>\epsilon H_0{}^4M^{-1}$ for sufficiently large u, which gives a contradiction. Therefore $r'_\gamma(u)\to 2M$ as $u\to+\infty$, i.e. γ approaches the corresponding generator asymptotically.

To prove the uniqueness we observe that around every point of $\mathcal{M}$, n can locally be transformed to the form (2.18), (2.19) by means of a transformation (2.15)–(2.17). The uniqueness of *bounded* and *everywhere* defined solutions of (2.16) means, however, that any integral curve of n other than those generating $\mathcal{N}$ must either reach $r=0$ or go to infinity, $r\to+\infty$, as $u\to-\infty$.

Finally, $\mathcal{N}$ is given by

$$r=2MH\psi \tag{2.22}$$

in the original coordinates. Since M is constant we find

$$g^{ab}(r-2MH\psi)_{,a}(r-2MH\psi)_{,b}|_{\mathcal{N}}=-2\psi^{-2}\{[\partial_\theta(H\psi)]^2+\sin^{-2}\theta[\partial_\phi(H\psi)]^2\}, \tag{2.23}$$

which means that $\mathcal{N}$ is non-space-like. □

Remarks. In the special case of spherical symmetry $\mathcal{N}$ is the (time-reversed) event horizon, and in the general case $\mathcal{N}$ acts as a horizon with respect to radially directed causal curves. Being time-like at most points, it resembles the stationary limit surface of the Kerr solution (Hawking and Ellis 1974, p 165). Also note that the bounded r coordinate means that $\mathcal{N}$ is limited in the physical sense that the intersections $\mathcal{N}\cap\{u=\text{constant}\}$ have bounded area.

3. A class of cosmic time functions

Let $\mathcal{M}$ be a space–time as specified in § 2. If t is a CTF in $\mathcal{M}$ then each point of $\mathcal{N}$ lies in some of the PCS's $\{t=\text{constant}\}$, and each PCS is intersected in at most one point by each generator of $\mathcal{N}$. Our aim, to be reached in the next section, is to formulate sufficient conditions for PCS's in which the curvature singularity appears as a point. It will then be essential that each PCS *does* intersect $\mathcal{N}$.

As a preliminary, we demonstrate in this section an example of CTF's, whose level surfaces have the mentioned properties, namely

$$t=u-\kappa e^{-r} \tag{3.1}$$

where κ is a positive constant. This function is globally defined and

$$g^{ab}t_{,a}t_{,b}=2\kappa\, e^{-r}\left[1+\kappa\, e^{-r}\left(-K+\frac{\dot P}{P}r+\frac{M}{r}\right)\right] \tag{3.2}$$

which is positive for sufficiently small κ, whence t has everywhere a time-like gradient.

In the PCS's $\mathcal{H}_0=\{t=t_0\}$ the curvature singularity appears as a point in the following sense. $\mathcal{H}_0$ may be divided into two disconnected components, one of which is generated by curves incomplete at $r=0$, by an arbitrarily small topological two-sphere. Thus, t belongs to the class of CTF's with respect to which the curvature singularity may represent a mass point.

Further, the $\mathscr{H}_0$ are connected and they intersect $\mathcal{N}$. These two properties are, however, not sufficient to guarantee the 'point property' of the $\mathscr{H}_0$. We will therefore formulate three sufficient conditions on PCS's, no two of which are alone sufficient.

4. Properties of RT curvature singularities

In this section we select three conditions on PCS's in the RT space–times considered above. Then we show, in theorem 2, that in any such PCS the curvature singularity appears as a point. The corresponding choice of CTF is then a natural one in connection with particles represented by space–time singularities.

The conditions on any PCS $\mathscr{H}$ will be as follows.

(iv) $\mathscr{H}$ is connected;

(v) $\mathscr{H} \cap \mathcal{N} \neq \varnothing$;

(vi) there exist finitely many numbers u_0 such that $\lim u = u_0$ as $r \to 0$ along some curve in $\mathscr{H}$, and arbitrarily near each u_0 there are numbers u such that $\mathscr{H} \cap \mathcal{N}(u)$ is a topological two-sphere, where $\mathcal{N}(u)$ denotes the set of all integral curves of n starting at $(r = 0, u)$.

These conditions, although not necessary for theorem 2, are independent since the theorem does *not* follow from (iv) + (v), (iv) + (vi) or (v) + (vi). Condition (vi) is a precise statement of the idea that the singularity appears at a number of places *in* $\mathscr{H}$. However, we need not assume more since (iv) and (v) act as 'boundary conditions' forcing $\mathscr{H}$ to reach the singularity at just *one* u-value. The *a priori* reasons for the three conditions, i.e. besides the contents of the following theorem, will be discussed in § 5.

Theorem 2. Let $\mathcal{M}$ be an RT space–time as specified in theorem 1, and let $\mathscr{H}$ be a PCS in $\mathcal{M}$ satisfying conditions (iv)–(vi). Then $\mathscr{H}$ is divided into two parts, one of which is incomplete at $r = 0$, by an arbitrarily small topological two-sphere.

Proof. The proof is divided into three parts.

(I) First we show that there is only *one* u_0 as in condition (vi). From theorem 1 and (v), $\mathscr{H} \cap \mathcal{N}(u) \neq \varnothing$ for all sufficiently large negative u. Thus, by (iv), if $\mathscr{H} \cap \mathcal{N}(\hat{u}) = \varnothing$ then $\mathscr{H} \cap \mathcal{N}(u) = \varnothing$ for all $u \geqslant \hat{u}$, since each $\mathcal{N}(u)$ divides $\mathcal{M}$ into two parts. Further, if $\lim u = u_0$ as $r \to 0$ along some curve in $\mathscr{H}$ then, from (vi), $\mathscr{H} \cap \mathcal{N}(u_0) = \varnothing$ since otherwise (by a small variation) some integral curve of n would intersect $\mathscr{H}$ in two points.

Let $\hat{u}_0$ be the smallest u_0 of (vi). Then $\mathscr{H} \cap \mathcal{N}(\hat{u}_0) = \varnothing$ and thus $\mathscr{H} \cap \mathcal{N}(u) = \varnothing$ for all $u \geqslant \hat{u}_0$, and we conclude that only *one* u_0 exists.

(II) Secondly, consider a topological two-sphere $\mathscr{T}(u) = \mathscr{H} \cap \mathcal{N}(u) \neq \varnothing$ with u in some arbitrarily small interval $0 < u_0 - u < \epsilon$. Since $\mathscr{H}$ is a connected (iv) PCS and $\mathcal{N}(u)$ divides $\mathcal{M}$ into two parts, $\mathscr{T}(u)$ divides $\mathscr{H}$ into two parts, precisely *one* of which is incomplete at $r = 0$ since only one u_0 of (vi) exists.

(III) Finally, we prove that with suitable u the upper limit of distances on $\mathscr{T}(u)$ can be made arbitrarily small. Being a PCS, $\mathscr{H}$ is globally given by $r = f(u, \theta, \phi)$. Then a time-like coordinate can be introduced by

$$x'^4 = r - f \tag{4.1}$$

with $x'^4 = 0$ on $\mathscr{H}$. In a surrounding of $\mathscr{H}$ a space-like coordinate r' can be defined, which is constant on each $\mathscr{T}(u)$, and $r' \to 0$ as $u \to u_0$. Corresponding to (r', θ, ϕ, x'^4)

there is a reference system in which we calculate distances for $x'^4 = 0$, i.e. in $\mathscr{H}$. If g'_{ab} denote the components of the metric tensor in these coordinates then the line element in $\mathscr{H}$ is given by a three-tensor $\mathbf{h}'$ with components

$$h'_{\alpha\beta} = g'_{\alpha\beta} - g'_{\alpha 4} g'_{\beta 4} / g'_{44} \tag{4.2}$$

and inverse $g'^{\alpha\beta}$.

Now,

$$\frac{\partial r'}{\partial x^A} \to 0 \qquad \text{as } u \to u_0 \tag{4.3}$$

$(A = 2, 3)$. Therefore, near the singularity, the g'^{1A} are insignificant compared with $g'^{AB} = g^{AB}$. Further, $\mathscr{T}(u)$ can be given by $r = f_{\mathscr{T}}(\theta, \phi)$ where $f_{\mathscr{T}}$ is a continuous function on S^2 and $\sup f_{\mathscr{T}} \to 0$ as $u \to u_0$. Since r' is constant on $\mathscr{T}(u)$, the line element on $\mathscr{T}(u)$ is essentially given by

$$d\sigma^2 = -g_{AB}\, dx^A\, dx^B = \tfrac{1}{2} f_{\mathscr{T}}^2 H^{-2}\, d\Omega^2. \tag{4.4}$$

Since H^{-2} is bounded, it is clear that the shortest distance between any points on $\mathscr{T}(u)$ goes to zero as $u \to u_0$. □

5. Conclusion

For a certain class of RT space–times we have given examples of PCS's in which the curvature singularity appears as a point, and we have found sufficient conditions (iv)–(vi) for a PCS to have this property. On the other hand, there also exist PCS's not possessing the one-point property, for instance the following ones in the Finkelstein case ($K = \frac{1}{2}$, $H = 1/\sqrt{2}$, $M = \text{constant} > 0$).

1. In the PCS's $\{r = r_0 < 2M\}$ the singularity does not appear at all, i.e. these PCS's contain no incomplete curves.

2. In the PCS's $\{r + (r+u)^2/M = r_0 < 2M\}$ the singularity appears as two points, at $u = \pm\sqrt{r_0 M}$.

3. Let $F_{t_0}(r, u) = r - M - M^2(u + r - t_0)^{-1}$. For any t_0, $\{F_{t_0}(r, u) = 0\}$ consists of two PCS's separated by $r = M$, in one of which the singularity appears as a point and in the other does not appear at all. Together with $\{r = M\}$ these PCS's cover $\mathscr{M}$. A corresponding CTF is defined by

$$t = \begin{cases} F_-[u + r - M^2/(r - M)] & r < M \\ 0 & r = M \\ F_+[u + r - M^2/(r - M)] & r > M \end{cases} \tag{5.1}$$

where F_+ and F_- are suitable strictly increasing functions from R onto R_+ and R_-, respectively. Thus, the singularity appears as a point for $t < 0$, vanishes at $t = 0$, and then remains absent.

Examples, such as the above, show that singularities may represent particles only with respect to certain CTF's. It is also clear that the situation is caused by the *horizon* rather than the singularity as such. Therefore, qualitatively the same thing will happen even if particles are not identified with mass points provided only that they are small enough to lie inside a horizon. In the case of elementary particles this may not be so, but this kind of situation may occur in gravitational collapse, as soon as a black hole has formed, even if the matter does not contract to infinite density. Also, if singularities or very dense states of matter are always surrounded by an event horizon (the 'cosmic censorship' hypothesis), then essentially the same thing happens with other singular space–times than the RT ones. Only if horizons could be excluded from space–time as unphysical would the situation change (Einstein 1939, Rosen 1970, Møller 1978). However, due to the global investigations of space–time manifolds satisfying the Einstein field equations, this does not seem feasible and, in any case, it is not the majority's view (Hawking and Ellis 1974 and references therein). Thus, at least within general relativity, some selection of CTF's or PCS's is necessary.

We can now also see some *a priori* reasons for the choice of conditions (iv)–(vi). The conditions should be selected so as to avoid the kind of PCS's made possible by the existence of an event horizon. Demanding a non-empty intersection with a horizon excludes the kind of PCS's in example 2. The same thing happens with our hypersurface $\mathcal{N}$, being generated by null curves (v). Also examples 1 and 3, which are excluded by (vi), are made possible by the null cone structure, typical of the surrounding of a horizon. The possibility of violating condition (iv) does not, however, depend on the horizon but rather on the incompleteness of space–time at a singularity.

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